JOURNAL OF COMBINATORIAL THEORY, Series A **69**, 170–172 (1995)

Note

Some New Difference Sets

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Communicated by William M. Kantor

Received June 8, 1993

Existence status of (96, 20, 4) difference sets in $Z_4 \times Z_4 \times Z_2 \times Z_3$ has been open so far. In this paper we construct these difference sets, thereby filling a missing entry in Lander's table with the answer "yes," © 1995 Academic Press, Inc.

1. INTRODUCTION

Let G be a multiplicative group of order v. A subset D of G of size k is said to be a (v, k, λ) difference set in G if the differences $d(d')^{-1}$ for d, $d' \in D$ contain every nonidentity element of G exactly λ times. We say that D is abelian (resp. cyclic) if G is abelian (resp. cyclic). For more on difference sets, refer to Beth *et al.* [3] or Lander [6]. For recent results, refer to Jungnickel [5].

In this paper we study (96, 20, 4) abelian difference sets. Results of Turyn [8] and Lander [6] imply that the underlying abelian group can have exponent at most 24. McFarland [7] constructed (96, 20, 4) difference sets in $Z_2^5 \times Z_3$ and Dillon [4] constructed them in $Z_2^3 \times Z_4 \times Z_3$. Arasu and Sehgal [1] gave the first theoretical proof of the nonexistance of (96, 20, 4) difference sets in $Z_4 \times Z_8 \times Z_3$. Arasu *et al.* [2] prove (96, 20, 4) difference sets do not exist in either $Z_4 \times Z_8 \times Z_3$ or $Z_2 \times Z_2 \times Z_8 \times Z_3$.

*This work is partially supported by NSA Grant MDA 904-92-H-3057 and by NSF Grant NCR-9200265.

0097-3165/95 $6.00

Thus the only undecided abelian group is $Z_4 \times Z_4 \times Z_2 \times Z_3$. In this paper we construct $(96, 20, 4)$ difference sets in this group.

Difference sets are often studied in the context of a group ring ZG. It is easy to see that D is a (v, k, λ) difference set in G if and only if $DD^{(-1)} = (k - \lambda) + \lambda G$, where we identify the subset D of G with the group ring element $D = \Sigma_{d \in D} d$ and $D^{(-1)} = \Sigma_{d \in D} d^{-1}$. Each subset S of G is identified with $\Sigma_{x \in S} x$.

We make use of character theoretic results. When the underlying group is abelian, a character of the group is simply a homomorphism from the group to the multiplicative group of complex roots of unity. Extending this homomorphism to the entire group ring yields a map from the group ring to the complex numbers. The element D of ZG is a (v, k, λ) difference set in G if and only if

$$|\chi(D)| = \begin{cases} k & \text{if } x \text{ is the principal character} \\ \sqrt{k - \lambda} & \text{otherwise.} \end{cases}$$

2. Construction

Let $G = Z_4 \times Z_4 \times Z_2 \times Z_3 = \langle g_1 \rangle \times \langle g_2 > \langle g_3 \rangle \times \langle a \rangle$ (written multiplicatively). We work in ZG.

Let

$$A = 1 + g_1^2 + g_2^2 + g_1^2 g_2^2$$
$$B = 1 + g_1 + g_1 g_3 + g_1^2 g_3 + g_1^2 g_2 g_3 + g_1 g_2 + g_1 g_2^3 g_3 + g_2^3$$
$$C = 1 + g_1^3 + g_1^3 g_2^2 g_3 + g_1^2 g_2^2 g_3 + g_2 g_3 + g_1 g_2^3 g_3 + g_1 g_2^3 + g_1^2 g_2.$$

Proposition. *For arbitrary elements $x, y \in \langle g_1, g_2, g_3 \rangle$, $D = A + axB + a^2 yC$ is a $(96, 20, 4)$ difference set in G.*

Proof. Let $H = \langle g_1, g_2, g_3 \rangle$. We enumerate the 31 nonprincipal characters of H:

$\chi_1, \chi_2, \ldots, \chi_7$ are the 7 real characters

$\chi_8, \chi_9, \ldots, \chi_{19}, \bar{\chi}_8, \bar{\chi}_9, \ldots, \overline{\chi_{19}}$ are the 24 nonreal characters (Here: bar denotes complex conjunction). We define the first 19 characters above in the following table.

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}
g_1	-1	-1	-1	-1	1	1	1	i	i	i	i	i	i	i	i	1	1	-1	-1
g_2	1	1	-1	-1	1	-1	-1	1	1	-1	-1	i	i	$-i$	$-i$	i	i	i	i
g_3	1	-1	1	-1	-1	1	-1	1	-1	-1	1	1	-1	1	-1	1	-1	1	-1

Now for each of these 19 characters, we tabulate the character values $\chi_i(A), \chi_i(B), \chi_i(C)$, $1 \leq i \leq 19$.

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}
A	4	4	4	4	4	4	4	0	0	0	0	0	0	0	0	0	0	0	0
B	0	0	0	0	0	0	0	$4i$	4	0	0	0	0	$4i$	4	4	0	0	$-4i$
C	0	0	0	0	0	0	0	0	0	4	$-4i$	4	$-4i$	0	0	0	4	$4i$	0

Now for any nonprincipal character χ of G, the following two cases arise.

Case I. $\chi|_H$ is principal.

Then $\chi(D) = 4 + 8w + 8w^2$, where w is a primitive cube root of unity. Thus $\chi(D) = 4 - 8 = -4$, hence $|\chi(D)| = 4$.

Case II. $\chi|_H$ is nonprincipal.

From the second table, we see that exactly one of $\chi|_H(A)$, $\chi|_H(B)$ and $\chi|_H\,(C)$ is of modulus 4 and the other two are zero. Hence $\chi(D) = 4\xi$, where ξ is a 12th root of unity. Thus $|\chi(D)| = 4$.

Since $|\chi(D)| = 4$ for all nonprincipal characters χ and G, and $|D| = 20$, it follows that D is a $(96, 20, 4)$ difference set in G.

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